Closed-Form Expressions for the Current or Charge Distribution on Parallel Strips or Microstrip

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Abstract—**Simple but accurate closed-form expressions for the charge and current distributions on parallel-plate stripline or microstrip are given. This form is convenient for use in various applications, such as determining radiation or mode fields of the lines, or frequency dispersion of the fundamental mode of the microstrip. These expressions are used to obtain explicit expressions for the capacitance of these lines, accurate to within one percent of the actual value for any ratio of stripwidth to spacing.**

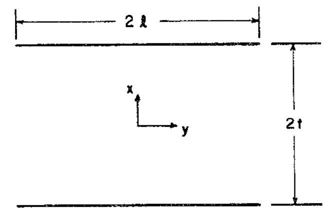

Fig. 1. Parallel-plate transmission line.

I. Introduction

THE STATIC capacitance per unit length of two infinite parallel perfectly conducting strips (see Fig. 1) of width $2l$ and separated by a distance $2t$ has an exact solution via the method of conformal transformation which has been extensively studied in the literature [1]–[7]. As with many conformal mapping results, however, neither the fields nor the capacitance are obtained explicitly, but require solution of one or more implicit equations involving incomplete elliptic integrals. The authors recently were in need of a reasonably simple, approximate but explicit expression for the charge distribution on one of the strips in an investigation of dispersion on wide microstrip [8]. Although a number of approximations (some of them only implicitly defined) for the capacitance can be found in the literature [1], [2], [9]–[15], only [15] gives an explicit approximation for the charge distribution, and it is rather cumbersome, requiring the reading of a number of constants from a graph.

It is known (see, e.g., [16]) that a *symmetric* stripline—a strip of width $2l$ located midway between and parallel to two infinite conducting planes which are separated by a distance $2h$—has an exact solution for charge (or current) distribution of the form

$$\rho(y)=\frac{\rho_0}{\sqrt{\cosh^2\left(\frac{\pi l}{2h}\right)-\cosh^2\left(\frac{\pi y}{2h}\right)}},\qquad |y|\leqslant l. \quad (1)$$

Depending on the ratio l/h and the constant ρ_0, this function can represent a variety of behaviors, but always satisfies the edge condition as $y\rightarrow\pm l$. The authors thus tried various values of h in an attempt to approximate the charge distribution on a parallel-plate stripline, and found (quite by accident) that the value $h=2t$ provided a very good fit for both narrow and wide strips. Since (1) is so simple, and may be useful in other applications such as calculating radiation and mode field distributions, it seemed worthwhile to attempt to give a more rigorous justification for it. This is done in the present paper, and is generalized to the case of a microstrip with a substrate permittivity different from free space. As an application of these results, explicit expressions for the static capacitance per unit length of these transmission lines are obtained, which are accurate to less than one percent for all values of l/t and ϵ_r. Although in some cases this represents only a modest improvement over the results of Wheeler [11], [20], the expressions for the capacitance are intended primarily as an illustration of the application of the closed-form expressions for charge distribution, which are the main result of the paper and are believed to be new.

Manuscript received April 30, 1979; revised October 18, 1979. This research was supported by the Office of Naval Research under Contract N0014-76-C-0318, monitored by Dr. H. Mullaney.
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II. Approximate Kernel and Charge Distribution for Parallel Plates

If the plates of Fig. 1 are kept at a potential difference V, standard Green's function techniques convert the field equations to an integral equation for $\rho(y)$ on one of the strips:

$$\frac{1}{\pi}\int_{-l}^{l}\rho(y')\ln\frac{\sqrt{(y-y')^2+4t^2}}{|y-y'|}\,dy'=V. \quad (2)$$

The capacitance C_p (normalized to the permittivity of the surrounding medium) is then given by

$$C_p=\frac{1}{V}\int_{-l}^{l}\rho(y)\,dy. \quad (3)$$

We seek to extract from the kernel of (2) a dominant singular part which is solvable in closed form, but which

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also behaves appropriately in both the wide and narrow strip limits.

The narrow strip limit is satisfied by any function which reduces to $-\ln|y-y'|$ plus some constant as $|y-y'|\to 0$. Taking $-\ln\left[\tanh\frac{\pi|y-y'|}{4h}\right]$ as a model (the solution of (2) with this kernel is simply (1); this is the kernel for the previously mentioned symmetric strip line), and using the fact that for wide strips $\rho(y)$ will be nearly constant over most of the strip, we choose h so that

$$-\int_{-\infty}^{\infty}\ln\left[\tanh\frac{\pi|y-y'|}{4h}\right]dy'=\int_{-\infty}^{\infty}\ln\frac{\sqrt{(y-y')^2+4t^2}}{|y-y'|}\,dy' \tag{4}$$

and thus (2) will be approximately satisfied for wide strips as well. Equation (4) is easily evaluated and shows that $h=2t$, confirming our empirical observation. The choice of an approximate kernel by matching the singularities and the areas is similar to an idea of Koiter [27] and Carrier [28], who use the method for Wiener–Hopf-type problems and provide explicit error criteria.

The difference ΔG between our exact and approximate kernels is nonsingular:

$$\ln\frac{\sqrt{(y-y')^2+4t^2}}{|y-y'|}\equiv G(y-y')=G_0(y-y')+\Delta G(y-y') \tag{5}$$

where

$$G_0(y-y')=-\ln\left[\tanh\frac{\pi|y-y'|}{8t}\right]$$

and

$$\Delta G(y-y')=\ln\left\{\sqrt{(y-y')^2+4t^2}\,\frac{\tanh\frac{\pi|y-y'|}{8t}}{|y-y'|}\right\}.$$

A plot of G, G_0, and ΔG as a function of $|y-y'|/t$ is shown in Fig. 2, demonstrating the fit that has been achieved.

With $h=2t$ in (1), we obtain our approximate expression for $\rho(y)$—except for the constant ρ_0 which will be proportional to the voltage V and the capacitance C_p. We determine this constant by integrating (1) from $-l$ to l and requiring that the result (the total charge) be C_pV. We obtain

$$\rho(y)=\frac{\pi C_p V}{8tk'K(k)}\left[\cosh^2\left(\frac{\pi l}{4t}\right)-\cosh^2\left(\frac{\pi y}{4t}\right)\right]^{-1/2} \tag{6}$$

where $K(k)$ is the complete elliptic integral of the first kind,

$$k=\tanh\left(\frac{\pi l}{4t}\right);\quad k'=(1-k^2)^{1/2}. \tag{7}$$

Expression (6) was compared with the exact charge distribution as computed from the implicit relations in, e.g., [3].[1] The worst agreement was found for l/t between about 1.5 and 3; a comparison for $l/t=2.37$ is given in Fig. 3. The largest discrepancy is in the relatively small region near the edge of the strip; even here, it is only about 12 percent for $y/l=0.99$. The overall fit, however, is seen to be quite good. For $l/t\lesssim 1$ or $\gtrsim 4$, the error was found to be a few percent or less.

[1]Caution must be observed that the Jacobian zeta function, which appears in this mapping is accurately evaluated, especially for wide strips which correspond to modulus close to unity. For this comparison, the algorithm of [17] was used.

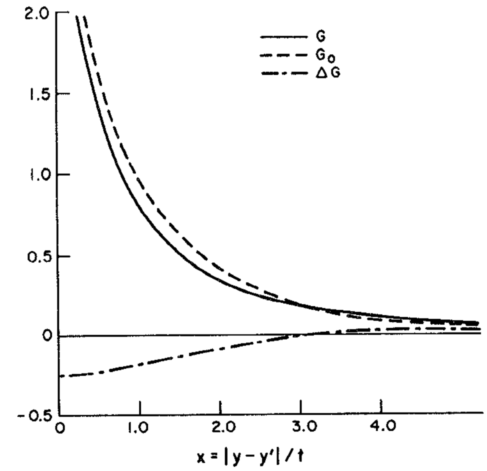

Fig. 2. Comparison of exact and approximate kernels (cf. (5)).

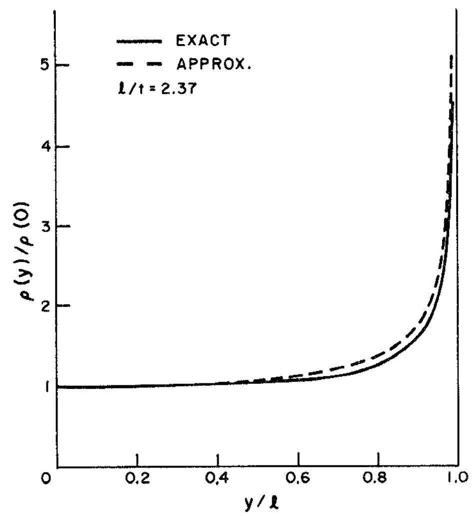

Fig. 3. Comparison of exact and approximate (11) charge distributions for parallel strips.

III. An Application: Closed-Form Expression for C_p

As in [18] and [16], we may construct a variational expression for the capacitance by multiplying (2) by $\rho(y)$ (now a trial function) and integrating from $-l$ to l. Using (3) and (5) we obtain

$$\frac{1}{C_p}=\frac{2}{C_s}+\frac{1}{\Delta C} \tag{8}$$

where

$$\frac{2}{C_s}=-\frac{\int_{-l}^{l}\int_{-l}^{l}\rho(y)\rho(y')\ln\left[\tanh\frac{\pi|y-y'|}{8t}\right]dy\,dy'}{\pi\left[\int_{-l}^{l}\rho(y)dy\right]^2} \tag{9}$$

and

$$\frac{1}{\Delta C}=\frac{\int_{-l}^{l}\int_{-l}^{l}\rho(y)\rho(y')\Delta G(y-y')\,dy\,dy'}{\pi\left[\int_{-l}^{l}\rho(y)\,dy\right]^2}. \tag{10}$$

If (6) is chosen for the trial function, then the term C_s in (8) can be identified as the capacitance per unit length of the aforementioned symmetric stripline; it can be evaluated exactly in terms of complete elliptic integrals of known modulus [16]:

$$C_s=4K(k)/K(k') \tag{11}$$

where k and k' are defined in (7). Although the correction term $1/\Delta C$ in (10) can apparently not be evaluated in closed form, we can be somewhat more careless with it since it is expected to be a small correction term. Thus we argue that for purposes of evaluating ΔC, $\rho(y)$ can simply be taken as a constant, because if $l/t \ll 1$, ΔG is nearly constant and the numerator of (10) approximately decouples into

$$\Delta G(0)\left[\int_{-l}^{l}\rho(y)\,dy\right]^2$$

so that from (10), $1/\Delta C \simeq \Delta G(0)/\pi$, which is very insensitive to what $\rho(y)$ really is. On the other hand, for wider strips, a constant distribution is closer to the actual distribution. Furthermore, ΔG itself is quite a well-behaved function, and can be excellently approximated (within a few percent over $0 \leqslant y < \infty$) by

$$\Delta G(y)\simeq 2\frac{(y/t)^2-a}{\left[(y/t)^2+a\right]^2};\quad a=\frac{2}{\ln(4/\pi)}\simeq 8.2794. \tag{12}$$

This function matches ΔG exactly at $y=0$ and asymptotically as $y\rightarrow\infty$, and integrates to zero from $y=-\infty$ to $+\infty$ as required of the exact ΔG. With these simplifications, the correction term is quite simple to calculate, and is found to be

$$\frac{1}{\Delta C}\simeq-\frac{t^2}{2\pi l^2}\ln\left[1+\frac{4l^2}{at^2}\right]. \tag{13}$$

Equations (8), (11), and (13) combine to give an *explicit* approximation to C_p:

$$\frac{1}{C_p}\simeq\frac{K\left(\operatorname{sech}\frac{\pi l}{4t}\right)}{2K\left(\tanh\frac{\pi l}{4t}\right)}-\frac{t^2}{2\pi l^2}\ln\left[1+\frac{4l^2}{at^2}\right]. \tag{14}$$

Computations based on (14) have been compared to exact results from the conformal mapping solution, and the largest error was again found in the region l/t between about 1.5 and 3.0. The largest error in C_p was found to be about 0.75 percent. This far exceeds the accuracy of most of the previously available closed-form approximations [1], [2], [9]–[11] and avoids the necessity of solving a transcendental equation as do the implicit approximations of [11]–[15]. The small price that is paid is the presence of elliptic integrals, but these are well tabulated and can be computed rapidly, even on small programmable calculators, using the arithmetic-geometric mean algorithm [19]. Wheeler [20] has recently given an analytical-empirical formula whose maximum error is a shade over 1 percent which involves only elementary functions, and which is probably preferable to (14) in most applications in view of the slight additional accuracy the latter provides. On the other hand, the derivation of (14) is less empirical and illustrates the use of the approximate charge distribution (6) and the approximate equivalence with a symmetric stripline.

IV. Approximate Charge Distribution and Capacitance for Microstrip

We can follow a similar line of argument for the static limit of the open microstrip illustrated in Fig. 4. Once again, by standard techniques (see, e.g., [16] or [18]), we find that the integral equation for the charge distribution on the strip (which is at a potential V with respect to the ground plane) is

$$\frac{1}{2\pi}\int_{-l}^{l}\rho(y')G_e^{(0)}(y-y')\,dy'=V \tag{15}$$

where the kernel $G_e^{(0)}(y)$ can be expressed as a Fourier integral:

$$G_e^{(0)}(y)=2\int_0^\infty\frac{\tanh\lambda}{\epsilon_r+\tanh\lambda}\cos\left(\lambda\frac{y}{t}\right)\frac{d\lambda}{\lambda} \tag{16}$$

ϵ_r is the relative permittivity of the substrate and t its thickness. The capacitance C_m of this microstrip, normalized to ϵ_0 is now

$$C_m=\frac{1}{V}\int_{-l}^{l}\rho(y)\,dy. \tag{17}$$

Now, the kernel for the symmetric stripline has a known Fourier integral representation [21, #4.116.2]:

$$-\ln\left[\tanh\frac{\pi|y|}{4h}\right]=\int_0^\infty\frac{\tanh\left(\lambda\frac{h}{t}\right)}{\lambda}\cos\left(\lambda\frac{y}{t}\right)d\lambda. \tag{18}$$

We wish to find some multiple of this function which matches the singular behavior of $G_e^{(0)}(y)$ as $y\rightarrow 0$, and which when integrated over $-\infty<y<\infty$ yields the same value as well. Elementary considerations from Fourier transform theory tell us that the first condition can be satisfied by adjusting the behavior of the integrand of (18) to match that of (16) for *large* λ, while the second is satisfied by matching the integrands at $\lambda=0$. From these two constraints we arrive at the approximate kernel

$$\begin{aligned}G_{e0}^{(0)}(y)&=\frac{2}{\epsilon_r+1}\int_0^\infty\frac{\tanh\left(\frac{\epsilon_r+1}{\epsilon_r}\lambda\right)}{\lambda}\cos\left(\lambda\frac{y}{t}\right)d\lambda\\&=-\frac{2}{\epsilon_r+1}\ln\left[\tanh\frac{\pi\epsilon_r|y|}{4(\epsilon_r+1)t}\right].\end{aligned} \tag{19}$$

The separation height h of the equivalent symmetric stripline is $(\epsilon_r+1)t/\epsilon_r$. We thus observe that the charge distribution for this microstrip should be approximately that

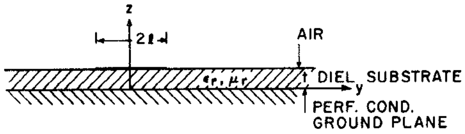

Fig. 4. Open microstrip.

corresponding to (19):

$$\rho(y)=\frac{\rho_0}{\sqrt{\cosh^2\left(\frac{\pi\epsilon_r l}{2(\epsilon_r+1)t}\right)-\cosh^2\left(\frac{\pi\epsilon_r y}{2(\epsilon_r+1)t}\right)}},\qquad |y|\leq l. \tag{20}$$

Since the total charge must be $C_m V$, we find, by an integration like that which led to (6), that

$$\rho_0=\frac{\pi\epsilon_r C_m V}{4(\epsilon_r+1)tk'_\epsilon K(k_\epsilon)} \tag{21}$$

where

$$k_\epsilon=\tanh\left(\frac{\pi\epsilon_r l}{2(\epsilon_r+1)t}\right);\quad k'_\epsilon=(1-k_\epsilon^2)^{1/2}. \tag{22}$$

Note that (20)–(21) reduce to (6) when $\epsilon_r\to 1$, as expected, while for $\epsilon_r\gg 1$, the charge distribution is nearly identical to that of a symmetric stripline with $h=t$. This approximate equivalence between different forms of stripline for large ϵ_r was apparently first noticed by Dukes [22], who argued that since most of the electric flux is concentrated in the substrate, the upper ground plane has only a small effect. Wheeler [23] used this limiting case as a partial basis for his original approximate conformal mapping solution for the static capacitance of the open microstrip.

By comparison of (20) with (6)—which would represent an approximation for the static *current* distribution for a microstrip with nonmagnetic substrate—it is seen that the charge distribution is somewhat "flatter," i.e., the charge decays faster away from the singularities at the edges of the strip than does the current. The effect is more pronounced the wider the strip, and is illustrated in Fig. 5 for a microstrip of $l/t=3$ and $\epsilon_r=10$. The effect of this difference in distributions is that a *transverse* current will be required on the strip for nonzero frequencies because of charge conservation. This current has been found to be an important contributor to the dispersion in wide microstrip [8].

By applying the variational method of Section III, we may come up with a closed-form approximation for C_m as well. As before, the part involving $G_{eo}^{(0)}$ gives rise to complete elliptic integrals, while a small correction term remains:

$$\frac{1}{C_m}=\frac{1}{2(\epsilon_r+1)}\frac{K(k'_\epsilon)}{K(k_\epsilon)}+\frac{\int_{-l}^{l}\int_{-l}^{l}\Delta G_e(y-y')\rho(y')\rho(y)\,dy\,dy'}{2\pi\left[\int_{-l}^{l}\rho(y)\,dy\right]^2}. \tag{23}$$

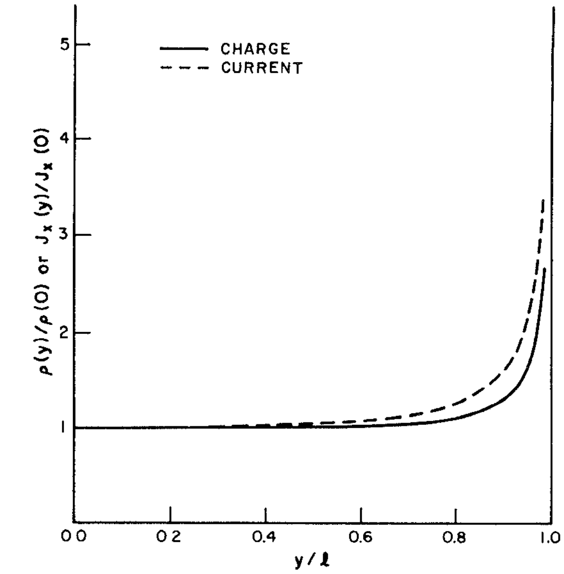

Fig. 5. Charge and current distributions on open microstrip $\epsilon_r=10$; $l/t=3$.

Here k_ϵ and k'_ϵ are defined by (22), while

$$\Delta G_e(y)\equiv G_e^{(0)}(y)-G_{eo}^{(0)}(y)=2\int_0^\infty\frac{d\lambda}{\lambda}\left[\frac{\tanh\lambda}{\epsilon_r+\tanh\lambda}-\frac{\tanh\left(\frac{\epsilon_r+1}{\epsilon_r}\lambda\right)}{\epsilon_r+1}\right]\cdot\cos\left(\lambda\frac{y}{t}\right). \tag{24}$$

We can again use rougher approximations to evaluate the correction terms, taking $\rho(y)$ as a constant, and approximating ΔG_e over $0<y<\infty$ by

$$\Delta G_e(y)\simeq\frac{2}{\epsilon_r^2}\frac{(y/t)^2-a_\epsilon}{\left[(y/t)^2+a_\epsilon\right]^2};$$
$$a_\epsilon=-\frac{(\epsilon_r+1)/\epsilon_r^2}{Q(-\delta_\epsilon)+\ln\left[\frac{\pi\epsilon_r}{2(\epsilon_r+1)}\right]} \tag{25}$$

where $\delta_\epsilon=(\epsilon_r-1)/(\epsilon_r+1)$, and the function

$$Q(x)\equiv\sum_{m=1}^{\infty}x^m\ln\left(\frac{m+1}{m}\right) \tag{26}$$

which has previously appeared in analyses of narrow microstrip [24] has been introduced. As with (12), the expression in (25) has been chosen to match ΔG_e exactly at $y=0$ and asymptotically as $y\to\infty$, and to integrate to zero from $y=-\infty$ to $+\infty$, as is required of the exact expression (24). The correction term is now easily calculated, and found to be

$$-\frac{t^2}{4\epsilon_r^2\pi l^2}\ln\left[1+\frac{4l^2}{a_\epsilon t^2}\right].$$

Thus we obtain an explicit approximation to C_m

$$\frac{1}{C_m} \simeq \frac{1}{2(\epsilon_r+1)}\frac{K(k'_\epsilon)}{K(k_\epsilon)} - \frac{t^2}{4\pi\epsilon_r^2 l^2}\ln\left[1+\frac{4l^2}{a_\epsilon t^2}\right]. \quad (27)$$

(Note that for $\epsilon_r=1$, $C_m=2C_p$, as found from (14), as we expect.) As $l/t \to 0$, this expression reduces to the one quoted in [24] for narrow strips; in the opposite limit, we find

$$C_m \simeq \epsilon_r \frac{2l}{t} + (\epsilon_r+1)\frac{4}{\pi}\ln 2 \quad (28)$$

which gives the correct parallel-plate value plus a smaller fringing term.

Testing the accuracy of (27) is somewhat more difficult because no strictly "exact" solution exists, and comparison must be made with numerical solutions of high accuracy. Moreover, since two parameters (l/t and ϵ_r) are now involved, checking accuracy over all ranges is more tedious. For the cases we tested, using $\epsilon_r \simeq 10$, the relative error was smaller than that for the same value of l/t but with $\epsilon_r=1$. In fact, the accuracy of (27) and (20) will increase with ϵ_r, because ΔG_e falls off essentially as ϵ_r^{-2}, while $G_e^{(0)}$ itself does so only as $1/\epsilon_r$. Thus the errors in (6) and (14) discussed in Section III will be upper bounds for those of (20) and (27) if $\epsilon_r > 1$.

We should emphasize that the current and charge distributions (6) and (20) are the principal results of this paper, and that (14) and (27) are intended as illustrative applications of these results (another application is found in [8]). The presence of the function $Q(-\delta_\epsilon)$ probably precludes the use of small programmable calculators to evaluate (27) at present, and Wheeler's [20] analytical-empirical formula (whose accuracy he estimates at 2 percent) should be adequate for such purposes. In addition, Wheeler's formula is reversible and can be used either for analysis or for synthesis. However, (27) is easily programmed on most ordinary computers and provides at least twice the accuracy of Wheeler's formula (if this is required) without the use of more time-consuming numerical procedures.

We might note in closing that the only other closed-form variational results for C_m of which we are aware are those of Vaynshteyn and Fialkovskii [25], [26]. These authors, however, obtain accuracy comparable to that of (27) only by using a function defined by a doubly infinite summation which can be quite tedious to evaluate numerically.

V. Conclusion

It has been shown that charge and current distributions on parallel-plate and microstrip transmission lines are approximated quite well by expressions (6) and (20) over the entire range of values of l/t and ϵ_r. Because of their simplicity, these functions should be useful in many applications where an accurate knowledge of these distributions is required. In addition to dispersion calculations [8], these might also include computing the mode fields of a parallel-plate transmission line, evaluating the radiation from microstrip discontinuities, or computation of the performance of stripline antennas. In this paper, they have been used to obtain extremely accurate closed-form expressions for the capacitance of parallel-plate and microstrip transmission lines, with an accuracy of a fraction of one percent. The primary usefulness of (6) and (20), however, should be in applications where numerous evaluations of charge and current density need to be made (e.g., when these distributions appear inside an integral), and considerable computer time is to be saved by using these instead of numerically obtained values.

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Theory of Dispersion in Microstrip of Arbitrary Width

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Abstract—**An analytic theory for the dispersion of the fundamental mode on wide open microstrip is presented. Only a single basis function is needed to accurately represent each of the change and current distributions on the strip, thus allowing more efficient determination of the propagation constant as compared to moment-method solutions requiring a larger number of basis functions. The results obtained blend smoothly into results of high-frequency (Wiener–Hopf) theories, and still retain the appealing physical interpretation in terms of capitance and inductance of the narrow strip theory previously obtained by the authors.**

I. Introduction

In PREVIOUS work, the authors [1] have presented an analytic theory of dispersion for narrow open microstrip (that is, for which the strip is small compared to substrate thickness) in terms of a dispersive series inductance and capitance, generalizing the classical expression for the propagation constant from transmission line theory which involves the static values of these parameters. Because an accurate form for the current and charge distributions (which are the same for this case) was available, it was possible to avoid more cumbersome moment function expansions, and to obtain a relatively simple dispersion relation possessing the clear physical interpretation referred to above. In reviewing numerical results available in the literature for wider microstrip, whose strip width is comparable to substrate thickness, the authors found significant discrepancies between workers who used different methods to attack the problem [2]. The best methods seem to be those which can represent the current and charge distributions (especially the edge singularities) accurately with a minimum number of basis functions.

The goal of the present study is to formulate an analytic theory of dispersion similar to [1] which will be valid for wider strips, yet still retain both analytical and computational straightforwardness as well as clear physical insight into the problem. Crucial to this is the recognition that the charge and current distributions now differ significantly from those in the narrow-strip limit, and also differ to some extent from each other. Thus an important part of the discussion depends on having accurate and reasonably simple functional descriptions of these distributions. The results will be examined to see what degree the difference of these distributions from the narrow-strip case and from each other affects the accuracy of the computed dispersion curves.

Of published numerical work, references [3]–[5] offer results that we might classify as applying to "wide" microstrip, and these will be used as the basis for comparison. Also, although we shall consider strips wide compared to the substrate, the strips are not allowed to become *electri-*

Manuscript received June 11, 1979; revised October 17, 1979. This project is supported in part by the Office of Naval Research under Contract N0014-76-C-0318 and in part by NSF Grant ENG78-09029.
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0018-9480/80/0300-0259$00.75 ©1980 IEEE